Reachability of quantified problems: from control to neural network global robustness

Éric Goubault, **Nuwan Herath Mudiyanselage**, Sylvie Putot

LIX, CNRS, École polytechnique, Institut Polytechnique de Paris, Palaiseau, France, UMR 7161

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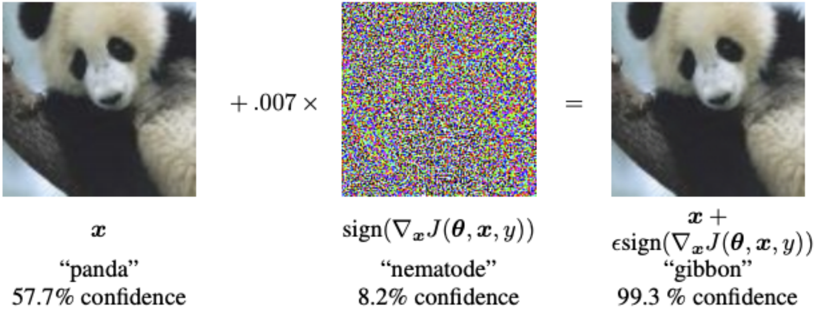
Sensitivity of a neural network



[Goodfellow et al., 2014]

The classification by neural networks can be sensitive to small perturbations
Can we formally verify that neural networks are robust?

Local robustness of a neural network

Let $N : \mathbb{R}^k \to \mathbb{R}^l$ be a neural network

Definition

We say that N classifies $\mathbf{p} \in \mathbb{R}^k$ as belonging to class $i \in [1, l]$ if for all $j \neq i$,

$$N_i(\mathbf{p}) > N_j(\mathbf{p})$$

We introduce the classical notion of local robustness of classification, which is checked at a given neighborhood of an input $\mathbf{p}$, for a bounded perturbation ϵ:

Definition

We say that N is ϵ-locally robust around $\mathbf{p} \in \mathbb{R}^k$, which is classified by class i if

$$\forall \mathbf{q} \in \mathbf{p} \oplus [-\epsilon, \epsilon]^k,\ N \text{ classifies } \mathbf{q} \text{ as belonging to class } i$$

Quantified problems

Quantified reachability problems are central in control and hybrid systems

A classical robust reachability problem consists in computing the states $\mathbf{z}$ reachable for some control $\mathbf{u}$ and initial state $\mathbf{x}_0$, independently of disturbances $\mathbf{w}$, within time $[0, T]$, for a given function ϕ

$$\mathcal{R} = \{\mathbf{z} \in \mathbb{R}^m \mid \exists \mathbf{u} \in \mathbb{U}, \exists \mathbf{x}_0 \in \mathbb{X}_0, \forall \mathbf{w} \in \mathbb{W}, \exists s \in [0, T], \mathbf{z} = \phi(s; \mathbf{x}_0, \mathbf{u}, \mathbf{w})\}$$

We study the inverse problem. We want now to characterize, for $\mathbb{G}$ a given target or goal region, the subset of the initial states $\mathbb{X}_0$ for which region $\mathbb{G}$ is robustly reachable:

$$\Sigma = \{\mathbf{x}_0 \in \mathbb{X}_0 \mid \exists \mathbf{u} \in \mathbb{U}, \forall \mathbf{w} \in \mathbb{W}, \exists s \in [0, T], \phi(s; \mathbf{x}_0, \mathbf{u}, \mathbf{w}) \in \mathbb{G}\}$$

Problem statement

Let f be a function from $\mathbb{R}^{n+2l}$ to $\mathbb{R}^m$. We want to characterize the set

$$\Sigma = \{\mathbf{x} \in \mathbb{D} \mid \forall \mathbf{p}_1 \in \mathbb{P}_1, \exists \mathbf{p}_2 \in \mathbb{P}_2, \ldots, \forall \mathbf{p}_{2l-1} \in \mathbb{P}_{2l-1}, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, f(\mathbf{x}, \mathbf{p}) \in \mathbb{G}\}$$

This covers any case: for example, for $\exists\forall\exists$ or $\forall\forall\exists$

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More generally

$$\Sigma = \{\mathbf{x} \in \mathbb{D} \mid \ldots, \bigvee_{i \in I} \bigwedge_{j \in J} f_{i,j}(\mathbf{x}, \mathbf{p}) \in \mathbb{G}_{i,j}\}$$

This covers any case: for example, for $\exists\forall\exists$ or $\forall\forall\exists$

A PREREQUISITE

Interval arithmetic

When dealing with a value v not known or not represented exactly, its range is bounded above and below:

$$v_l \leq v \leq v_u.$$

In interval arithmetic, we represent the inequalities with an interval:

$$v \in [v_l, v_u].$$

Interval arithmetic is used to compute the enclosure of sets of solutions to computational problems. Algorithms can rely on interval arithmetic to handle

- accumulated rounding errors,
- approximation errors,
- propagated uncertainties...

The lower and upper bounds of an interval I are denoted $\underline{I}$ and $\overline{I}$. So,

$$I = [\underline{I}, \overline{I}].$$

The arithmetic of interval arithmetic

For $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$,

- $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$
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Interval arithmetic is memoryless and ignores the dependence of repeated occurrences of a variable

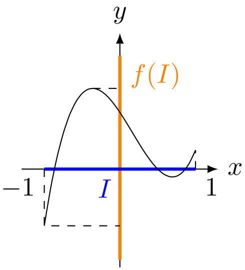
$$[0, 2] - [0, 2] = [-2, 2]$$

So $I - I$ is treated as $I - J$

Overestimation and convergence

The interval evaluation of I, $f(I)$, is an overestimation of the image set:

$$\{f(x) \mid x \in I\} \subseteq f(I)$$



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$$\lim_{[a,b]\to[x,x]} f([a,b]) = f(x)$$

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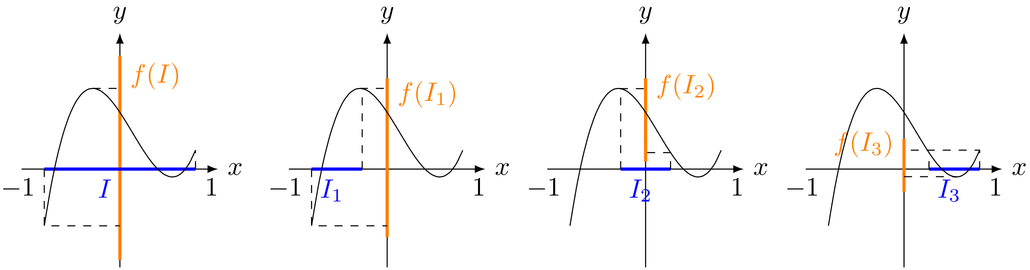

Convergence property $\longrightarrow$ subdivisions $I = I_1 \cup I_2 \cup I_3$

$$\lim_{[a,b]\to[x,x]} f([a,b]) = f(x)$$

SET PAVING

Set Inversion Via Interval Arithmetic

SIVIA algorithm illustration

$$\Sigma = \{\mathbf{x} \in \mathbb{D} \mid f(\mathbf{x}) \geq 0\}$$

$\mathcal{O}^{\mathsf{IN}}(\mathbb{X})$:

$$f(\mathbb{X}) \geq 0 \implies \mathbb{X} \subseteq \Sigma$$

$\mathcal{O}^{\mathsf{OUT}}(\mathbb{X})$:

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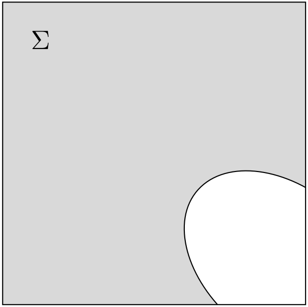

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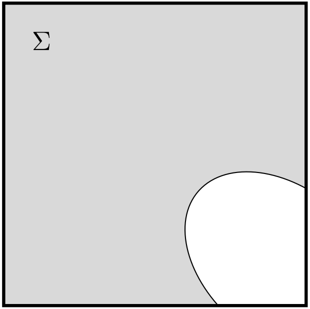

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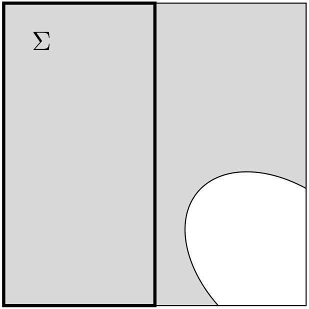

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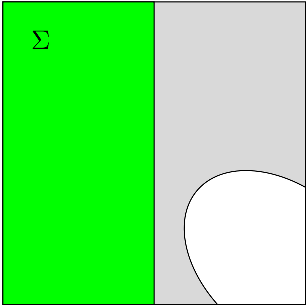

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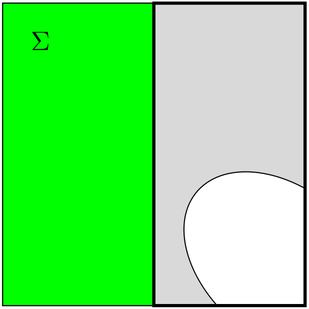

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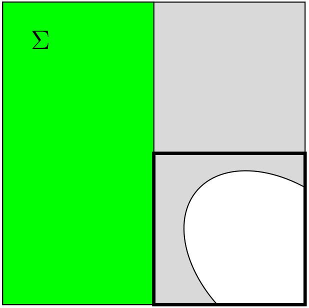

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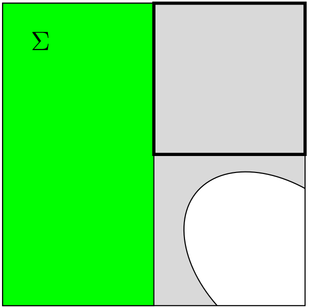

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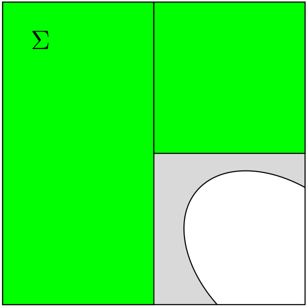

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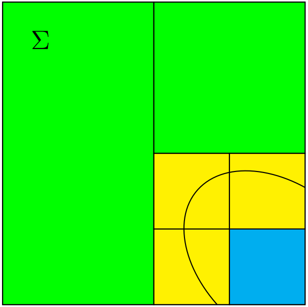

Quantified Set Inversion Via Interval Arithmetic

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This is true when $\mathbf{0}$ is in the reachable set

$$\mathcal{R}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^m \mid \forall \mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

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$$\boxed{\mathbf{0} \notin \mathcal{R}(\mathbb{X}^{\exists}, \mathbb{P}, \mathbb{G})} \implies \mathbb{X} \cap \Sigma = \emptyset$$

Quantified SIVIA oracles

	Using $\mathbb{P}$ and $\mathbb{G}$	Using $\neg\mathbb{P}$ and $\mathbb{G}^{\mathsf{C}}$
$\mathcal{O}^{\mathsf{IN}}$	$\mathbf{0} \in \mathcal{R}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G})$	$\mathbf{0} \notin \mathcal{R}(\mathbb{X}^{\exists}, \neg\mathbb{P}, \mathbb{G}^{\mathsf{C}})$
$\mathcal{O}^{\mathsf{OUT}}$	$\mathbf{0} \notin \mathcal{R}(\mathbb{X}^{\exists}, \mathbb{P}, \mathbb{G})$	$\mathbf{0} \in \mathcal{R}(\mathbb{X}^{\forall}, \neg\mathbb{P}, \mathbb{G}^{\mathsf{C}})$

$$\mathcal{R}(\mathbb{X}^{Q}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^m \mid Q\mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

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Can we compute $\mathcal{R}(\mathbb{X}^{Q}, \mathbb{P}, \mathbb{G})$?

QUANTIFIER ELIMINATION

Reachability of scalar-valued functions

Let $f : \mathbb{R}^{j_1} \times \ldots \mathbb{R}^{j_{2l}} \to \mathbb{R}$ be a scalar-valued affine function

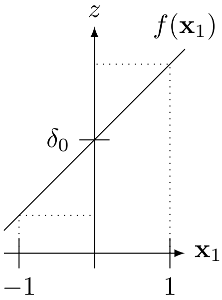
$$f(\mathbf{x}_1, \ldots, \mathbf{x}_{2l}) = \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \cdots + \langle \Delta_{2l}, \mathbf{x}_{2l} \rangle$$

with $\Delta_i \in \mathbb{R}^{j_i}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product

When all variables are existentially quantified

$$\mathcal{R}_\exists = \{z \mid \exists \mathbf{x}_1 \in [-1, 1]^{j_1}, z = \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle\}$$

$R_\exists = [\delta_0 - \|\Delta_1\|, \delta_0 + \|\Delta_1\|]$, where $\|\Delta_1\| = \sum_{k=1}^{j_1} |(\Delta_1)_k|$



Reachability of scalar-valued functions

Let $f : \mathbb{R}^{j_1} \times \ldots \mathbb{R}^{j_{2l}} \rightarrow \mathbb{R}$ be a scalar-valued affine function

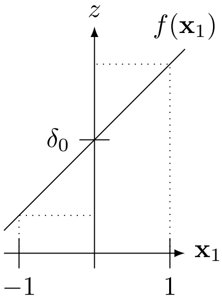
$$f(\mathbf{x}_1, \ldots, \mathbf{x}_{2l}) = \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \cdots + \langle \Delta_{2l}, \mathbf{x}_{2l} \rangle$$

with $\Delta_i \in \mathbb{R}^{j_i}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product

When all variables are universally quantified

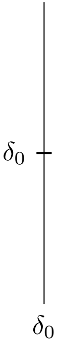
$$\mathcal{R}_\forall = \{z \mid \forall \mathbf{x}_1 \in [-1, 1]^{j_1}, z = \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle\}$$

If $\Delta_1 \neq \mathbf{0}$, $R_\forall = \emptyset$; otherwise $R_\forall(f) = [\delta_0, \delta_0]$



When there is a quantifier alternation

$$\mathcal{R}_{\forall\exists} = \{z \mid \forall x_1 \in [-1,1], \exists x_2 \in [-1,1], z = \delta_0 + \delta_1 x_1 + \delta_2 x_2\}$$



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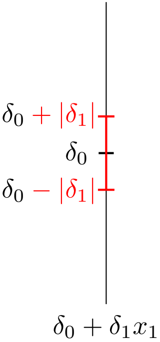

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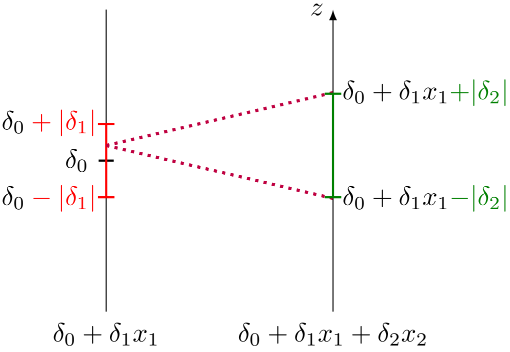

When there is a quantifier alternation

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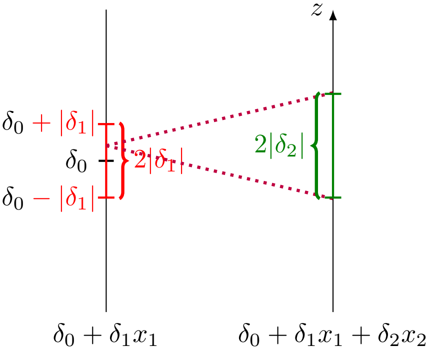

When there is a quantifier alternation

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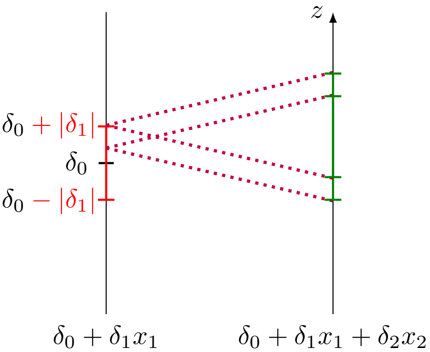

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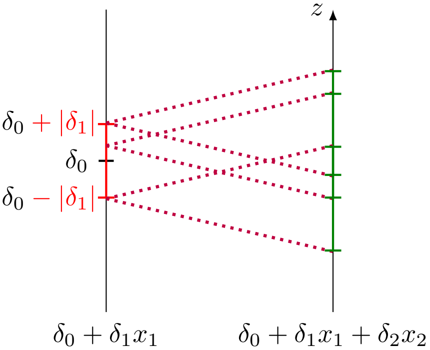

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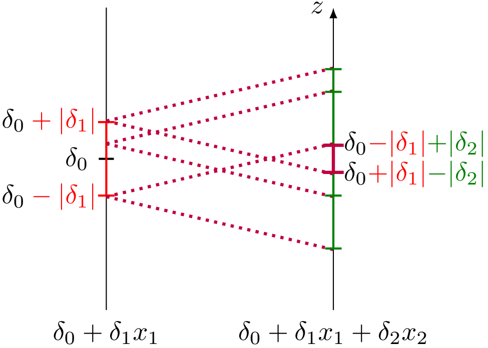

When there is a quantifier alternation

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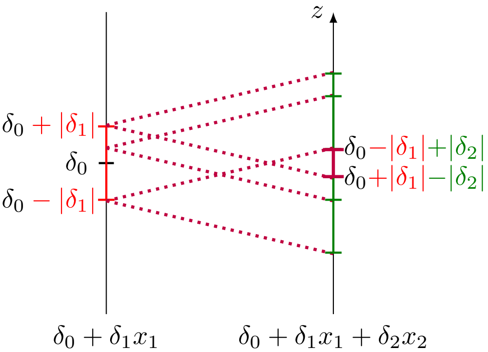
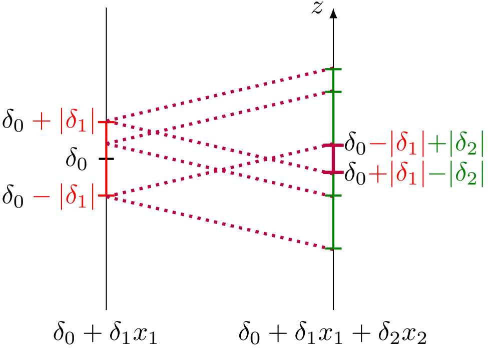


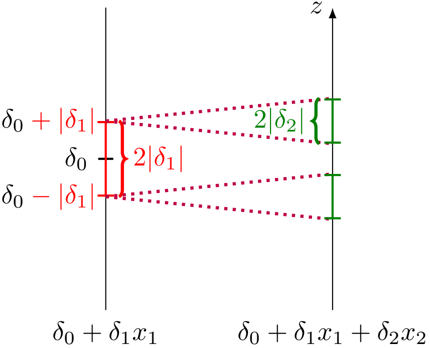
If $|\delta_1| \leq |\delta_2|$, then

$$R_{\forall\exists} = [\delta_0 + |\delta_1| - |\delta_2|, \delta_0 - |\delta_1| + |\delta_2|]$$

When there is a quantifier alternation

$$\mathcal{R}_{\forall\exists} = \{z \mid \forall x_1 \in [-1,1], \exists x_2 \in [-1,1], z = \delta_0 + \delta_1 x_1 + \delta_2 x_2\}$$





If $|\delta_1| \leq |\delta_2|$, then

$$R_{\forall\exists} = [\delta_0 + |\delta_1| - |\delta_2|, \delta_0 - |\delta_1| + |\delta_2|]$$

If $|\delta_1| > |\delta_2|$, then

$$R_{\forall\exists} = \emptyset$$

When there are l quantifier alternations

$$\mathcal{R} = \{\mathbf{z} \in \mathbb{R}^m \mid \forall \mathbf{x}_1 \in [-1,1]^{j_1}, \dots, \exists \mathbf{x}_{2l} \in [-1,1]^{j_{2l}}, \mathbf{z} = f(\mathbf{x}_1, \dots, \mathbf{x}_{2l})\}$$

When f is scalar-valued affine function, if for $j = 1, \dots, l$

$$\|\Delta_{2j-1}\| \leq \|\Delta_{2l}\| + \sum_{k=1}^{l} (\|\Delta_{2k}\| - \|\Delta_{2k-1}\|)$$

then

$$\mathcal{R} = \delta_0 + \left[\sum_{k=1}^{l} (\|\Delta_{2k-1}\| - \|\Delta_{2k}\|), \sum_{k=1}^{l} (\|\Delta_{2k}\| - \|\Delta_{2k-1}\|)\right]$$

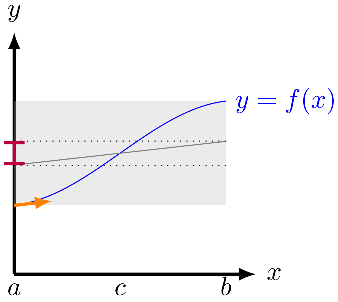
else

$$\mathcal{R} = \emptyset$$

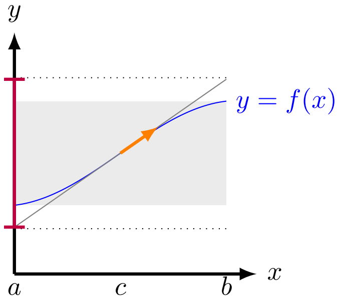
Function enclosure

Careful linearization of f to get an inner and an outer approximation of $\mathcal{R}$ using the mean value approximation

Let $I = [a, b]$ and $c = \frac{b-a}{2}$



Order 0 inner-approximation



Order 0 outer-approximation

$$y = f(c) + \underline{\nabla} \cdot (x - c)$$

$$y = f(c) + \overline{\nabla} \cdot (x - c)$$

$$\nabla = \left| \frac{df}{dx}(I) \right|$$

Inner and outer approximations of arbitrarily quantified reachability problems

[Goubault & Putot, 2025]
An interval-based method which allows for tractable but tight approximations thanks to careful linearization

$$\mathcal{R}^- \subseteq \mathcal{R} \subseteq \mathcal{R}^+$$

Alternative oracle implementations

	Using $\mathbb{P}$ and $\mathbb{G}$	Using $\neg\mathbb{P}$ and $\mathbb{G}^{\complement}$
$\mathcal{O}^{\mathsf{IN}}$	$\mathbf{0} \in \mathcal{R}^{-}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G})$	$\mathbf{0} \notin \mathcal{R}^{+}(\mathbb{X}^{\exists}, \neg\mathbb{P}, \mathbb{G}^{\complement})$
$\mathcal{O}^{\mathsf{OUT}}$	$\mathbf{0} \notin \mathcal{R}^{+}(\mathbb{X}^{\exists}, \mathbb{P}, \mathbb{G})$	$\mathbf{0} \in \mathcal{R}^{-}(\mathbb{X}^{\forall}, \neg\mathbb{P}, \mathbb{G}^{\complement})$

$$\mathcal{R}(\mathbb{X}^{Q}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^{m} \mid Q\mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

$$\mathcal{R}(\mathbb{X}^{Q}, \neg\mathbb{P}, \mathbb{G}^{\complement}) = \{\zeta \in \mathbb{R}^{m} \mid Q\mathbf{x} \in \mathbb{X}, \exists \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \forall \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}^{\complement}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

SIVIA algorithm for quantified problems

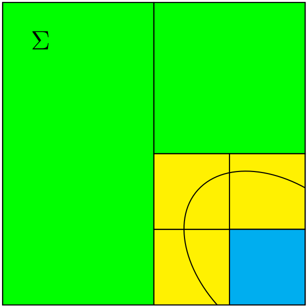
$$\Sigma = \{\mathbf{x} \in \mathbb{D} \mid \forall \mathbf{p}_1 \mathbb{P}_1, \exists \mathbf{p}_2 \in \mathbb{P}_2, \ldots, f(\mathbf{x}) \geq 0\}$$

$\mathcal{O}^{\mathsf{IN}}(\mathbb{X})$:

$$\mathbf{0} \in \mathcal{R}^{-}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G}) \implies \mathbb{X} \subseteq \Sigma$$

$\mathcal{O}^{\mathsf{OUT}}(\mathbb{X})$:

$$\mathbf{0} \notin \mathcal{R}^{+}(\mathbb{X}^{\exists}, \mathbb{P}, \mathbb{G}) \implies \mathbb{X} \subseteq \Sigma^{\complement}$$



Improvement by also subdividing $\mathbb{P}$

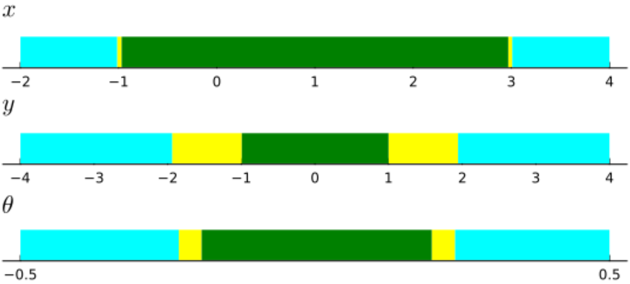
APPLICATIONS
TO CONTROL AND NEURAL NETWORKS

Dubins vehicle

$$\Sigma = \{\mathbf{x} \mid \exists \mathbf{x}_0 \in \mathbb{X}_0, \exists \mathbf{u} \in \mathbb{U}, \forall \mathbf{w} \in \mathbb{W}, \exists s \in [0, T], \mathbf{x} = \phi(s; \mathbf{x}_0, \mathbf{u}, \mathbf{w})\}$$

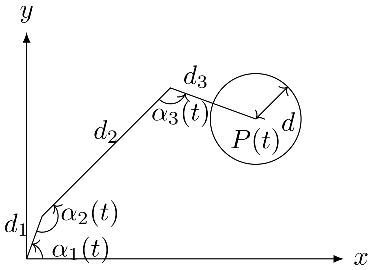
such that $\phi(s; \mathbf{x}_0, \mathbf{u}, \mathbf{w}) = (x, y, \theta)$ where

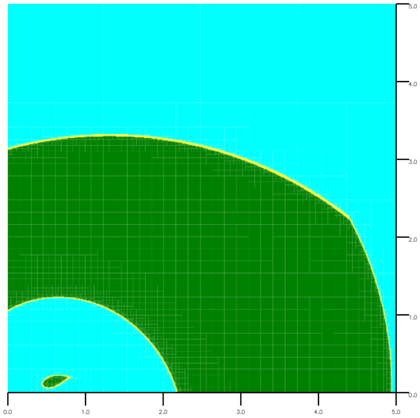
$$\dot{x} = v\cos(\theta) + \mathbf{w}$$
$$\dot{y} = v\sin(\theta)$$
$$\dot{\theta} = \mathbf{u}$$
$$v = 1$$



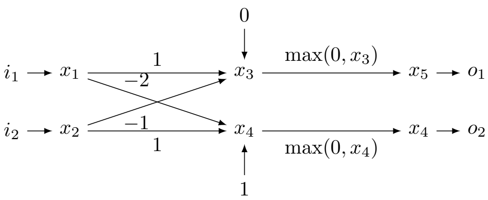
Robot collision problem

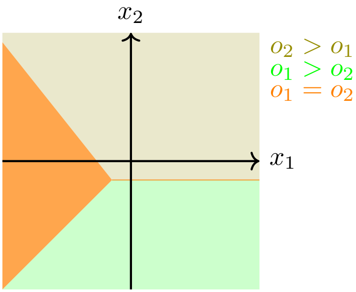
$$\Sigma = \left\{ (x,y) \mid \forall t \in [0,2],\ \sqrt{(x-P_x(t))^2 + (y-P_y(t))^2} \geq d \right\}$$





Application on a toy neural network

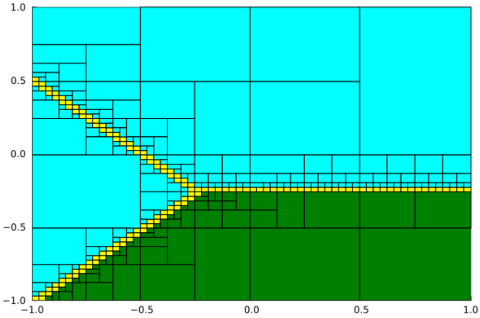




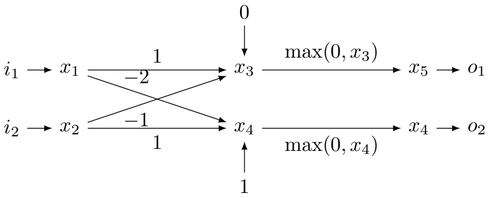
For the neural network $N : \mathbb{R}^2 \to \mathbb{R}^2$, we define the confidence in class 1 as

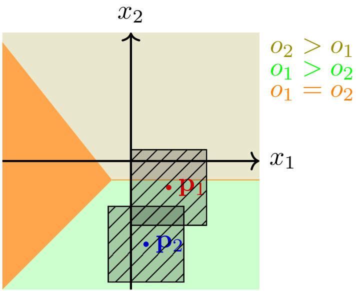
$$C_1(\mathbf{x}) = N(\mathbf{x})_1 - N(\mathbf{x})_2$$

$$\Sigma = \{\mathbf{x} \in \mathbb{D} \mid C_1(\mathbf{x}) > 0\}$$



Local robustness of a neural network





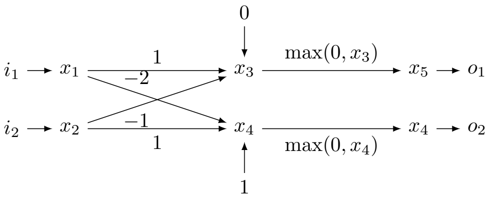
Not satisfied

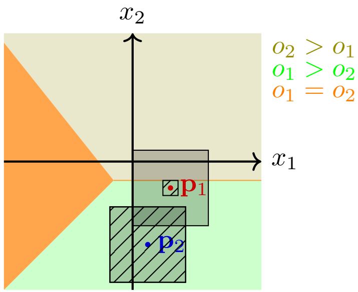
$$\forall \boldsymbol{\epsilon} \in [-\epsilon_{max}, \epsilon_{max}]^2,\ C_1(\mathbf{p}_1 + \boldsymbol{\epsilon}) > 0$$

Satisfied

$$\forall \boldsymbol{\epsilon} \in [-\epsilon_{max}, \epsilon_{max}]^2,\ C_1(\mathbf{p}_2 + \boldsymbol{\epsilon}) > 0$$

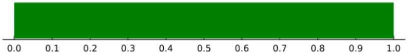
Local robustness of a neural network





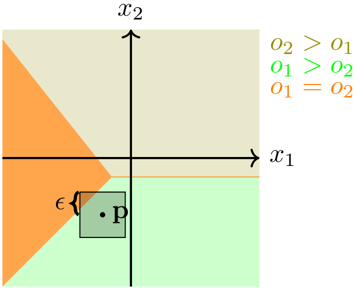
$$\Sigma = \{\lambda \in [0,1] \mid \forall \boldsymbol{\epsilon} \in [-\epsilon_{max}, \epsilon_{max}]^2,\ C_1(\mathbf{p} + \lambda\boldsymbol{\epsilon}) > 0\}$$

λ for $\mathbf{p}_1$

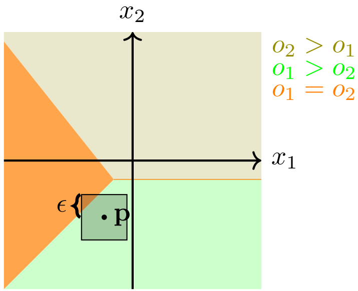


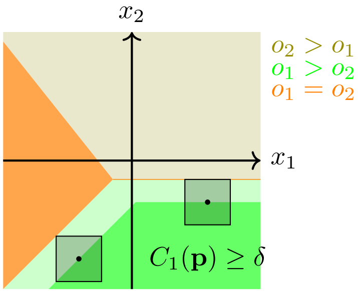
λ for $\mathbf{p}_2$

Global robustness for a class

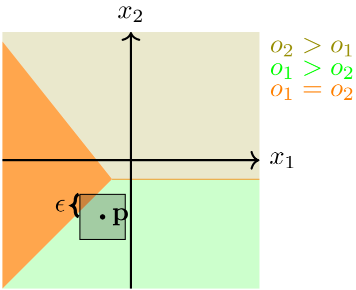


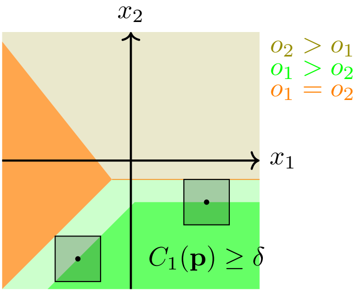
Global robustness for a class





Global robustness for a class





Definition

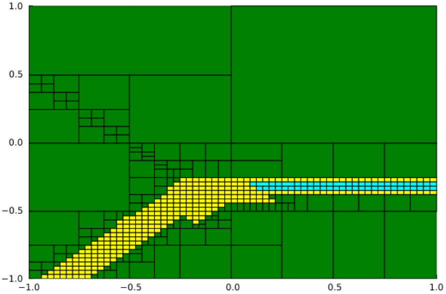
Consider a confidence function C_i for a class i for a neural network $N : \mathbb{R}^k \to \mathbb{R}^l$, a confidence level $\delta > 0$, a disturbance level $\epsilon > 0$. We say that N is (ϵ, δ)-globally robust for class i if

$$\forall \mathbf{p} \in \mathbb{R}^k,\ \left(C_i(\mathbf{p}) \geq \delta \Rightarrow \left(\forall \boldsymbol{\eta} \in [-\epsilon, \epsilon]^k,\ \mathbf{q} = \mathbf{p} + \boldsymbol{\eta} \text{ is classified in class } i\right)\right)$$

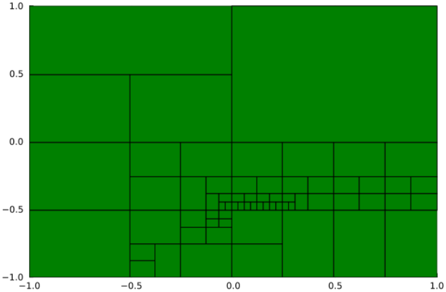
We take as confidence function $C_i : \mathbb{R}^k \to \mathbb{R}$: $C_i(\mathbf{p}) = \min_j\ (N_i(\mathbf{p}) - N_j(\mathbf{p}))_{1 \leq j \neq i \leq l}$

Global robustness of a neural network

$$\Sigma = \{\mathbf{x} \in [-1,1]^2 \mid \forall \boldsymbol{\epsilon} \in [-1/8, 1/8]^2,\ (C_1(\mathbf{x}) \geq \delta \implies C_1(\mathbf{x}+\boldsymbol{\epsilon}) > 0)\}$$



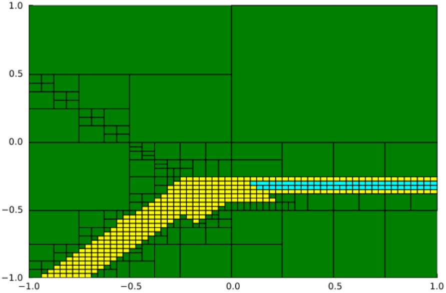
$\delta = 1/8$



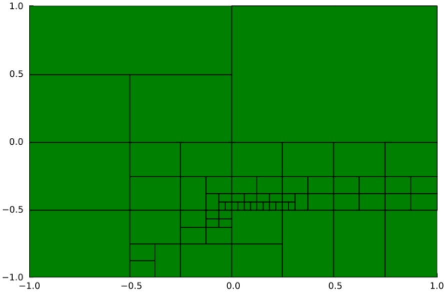
$\delta = 1$

Global robustness of a neural network

$$\Sigma = \{\mathbf{x} \in [-1,1]^2 \mid \forall \boldsymbol{\epsilon} \in [-1/8,1/8]^2,\ (C_1(\mathbf{x}) \geq \delta \implies C_1(\mathbf{x}+\boldsymbol{\epsilon}) > 0)\}$$

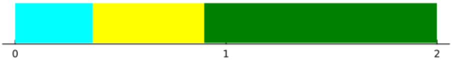


$\delta = 1/8$



$\delta = 1$

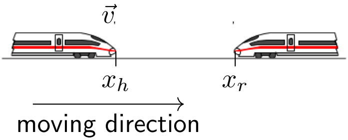
$$\Sigma = \{\delta \in \mathbb{R}_+ \mid \forall \mathbf{x} \in [-1,1]^2, \forall \boldsymbol{\epsilon} \in [-1/8,1/8]^2,\ C_1(\mathbf{x}) \geq \delta \implies C_1(\mathbf{x}+\boldsymbol{\epsilon}) > 0\}$$



European Train Control System

Two trains are running behind each other on one track into the same direction

The train behind shall be advised to initiate an emergency braking manoeuvre if it cannot stop without violating a safety distance to the train ahead



- v relative speed
- x_r rear position of the train ahead
- x_h head position of the train behind

[Grundt et al., 2022]

Neural network: 2 layers of 250 nodes with sigmoid activation functions

Is braking always advised when the trains are close?

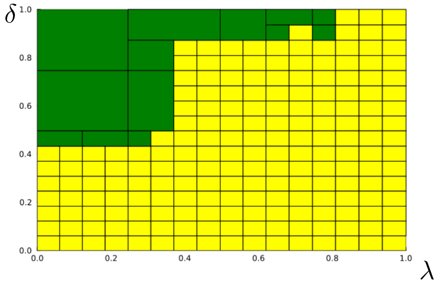
$$\Sigma = \{v \in [25, 149] \mid \forall x_r \in [0, 800], \forall x_h \in [0, 800], C_{brake}(v, x_h, x_r) > 0\}$$

$\Delta_{\mathbb{X}} = 0.01$ (computed in 7 min)

Optimizing global robustness

If the trains are 15 km apart, which confidence level δ and which perturbation level $\lambda\epsilon$, with $\epsilon = 1/8$, yield global robustness?

$$\Sigma = \{(\lambda, \delta) \in [0,1] \times [0,1] \mid \forall v \in [100, 149], \forall \epsilon \in [-1/8, 1/8],$$
$$C_{brake}(v, 20000, 35000) \geq \delta \implies C_{brake}(v + \lambda\epsilon, 20000, 35000) > 0\}$$



$\Delta_{\mathbb{X}} = 0.1, \Delta_{\mathbb{P}} = 0.01$ (computed in 30 min)

Ongoing and future work

- Explore other neural network properties (fairness, ...)
- Improve the algorithm
  - Better handling of more complex formulas like

$$\Sigma = \{\mathbf{x} \in \mathbb{D} \mid \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, M\}$$

    where

$$M = (f_1(\mathbf{x}, \mathbf{p}) \in \mathbb{G}_1 \wedge f_2(\mathbf{x}, \mathbf{p}) \in \mathbb{G}_2) \vee (\neg f_3(\mathbf{x}, \mathbf{p}) \in \mathbb{G}_3)$$

  - Adapt the refinement on $\mathbb{P}$
  - Use higher order approximation of f
- Investigate higher order formulas

Quantifiers and subdivisions

Let $\mathbb{P}_1 = \mathbb{A}_1 \cup \mathbb{B}$ and $\mathbb{P}_2 = \mathbb{A}_2 \cup \mathbb{B}$

$$[\forall \mathbf{p}_1 \in \mathbb{P}_1, \exists \mathbf{p}_2 \in \mathbb{A}_2, \Phi(\mathbf{p})] \vee [\forall \mathbf{p}_1 \in \mathbb{P}_1, \exists \mathbf{p}_2 \in \mathbb{B}_2, \Phi(\mathbf{p})] \implies \forall \mathbf{p}_1 \in \mathbb{P}_1, \exists \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})$$

$$[\exists \mathbf{p}_1 \in \mathbb{A}_1, \forall \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})] \vee [\exists \mathbf{p}_1 \in \mathbb{B}_1, \forall \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})] \implies \exists \mathbf{p}_1 \in \mathbb{P}_1, \forall \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})$$

$$[\forall \mathbf{p}_1 \in \mathbb{A}_1, \exists \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})] \wedge [\forall \mathbf{p}_1 \in \mathbb{B}_1, \exists \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})] \implies \forall \mathbf{p}_1 \in \mathbb{P}_1, \exists \mathbf{p}_2 \in \mathbb{P}_2, \Phi(\mathbf{p})$$

Refinement of $\mathbb{P}$

$$\mathcal{R}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^m \mid \forall \mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

If $\mathbb{P}_2 \supseteq \mathbb{Q}_2$,

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{Q}_2^{\exists}, \ldots, \mathbb{G})$$

So, if $\mathbb{P}_j = \bigcup_j \mathbb{P}_j^j$ and $\mathbb{P}_{2k}^{i_{2k}} \ni \check{\mathbb{P}}_{2k}^{i_{2k}}$

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \bigcup_{i_2} \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, (\mathbb{P}_2^{i_2})^{\exists}, \ldots, \mathbb{G})$$

Refinement of $\mathbb{P}$

$$\mathcal{R}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^m \mid \forall \mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

If $\mathbb{P}_2 \supseteq \mathbb{Q}_2$,

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If $\mathbb{P}_2 \ni \check{\mathbb{P}}_2$,

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \check{\mathbb{P}}_2^{\exists}, \ldots, \mathbb{G})$$

So, if $\mathbb{P}_j = \bigcup_j \mathbb{P}_j^j$ and $\mathbb{P}_{2k}^{i_{2k}} \ni \check{\mathbb{P}}_{2k}^{i_{2k}}$

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \bigcup_{i_2} \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, (\mathbb{P}_2^{i_2})^{\exists}, \ldots, \mathbb{G})$$

Refinement of $\mathbb{P}$

$$\mathcal{R}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^m \mid \forall \mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

If $\mathbb{P}_2 \supseteq \mathbb{Q}_2$,

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{Q}_2^{\exists}, \ldots, \mathbb{G})$$

If $\mathbb{P}_2 \ni \check{\mathbb{P}}_2$,

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \check{\mathbb{P}}_2^{\exists}, \ldots, \mathbb{G})$$

So, if $\mathbb{P}_j = \bigcup_j \mathbb{P}_j^j$ and $\mathbb{P}_{2k}^{i_{2k}} \ni \check{\mathbb{P}}_{2k}^{i_{2k}}$

$$\begin{aligned}\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) &\supseteq \bigcup_{i_2} \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, (\mathbb{P}_2^{i_2})^{\exists}, \ldots, \mathbb{G}) \\ &\supseteq \bigcup_{i_2} \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, (\check{\mathbb{P}}_2^{i_2})^{\exists}, \ldots, \mathbb{G})\end{aligned}$$

Refinement of $\mathbb{P}$

$$\mathcal{R}(\mathbb{X}^{\forall}, \mathbb{P}, \mathbb{G}) = \{\zeta \in \mathbb{R}^m \mid \forall \mathbf{x} \in \mathbb{X}, \forall \mathbf{p}_1 \in \mathbb{P}_1, \ldots, \exists \mathbf{p}_{2l} \in \mathbb{P}_{2l}, \exists \mathbf{z} \in \mathbb{G}, f(\mathbf{x}, \mathbf{p}) - \mathbf{z} = \zeta\}$$

If $\mathbb{P}_2 \supseteq \mathbb{Q}_2$,

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{Q}_2^{\exists}, \ldots, \mathbb{G})$$

If $\mathbb{P}_2 \ni \check{\mathbb{P}}_2$,

$$\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) \supseteq \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \check{\mathbb{P}}_2^{\exists}, \ldots, \mathbb{G})$$

So, if $\mathbb{P}_j = \bigcup_j \mathbb{P}_j^j$ and $\mathbb{P}_{2k}^{i_{2k}} \ni \check{\mathbb{P}}_{2k}^{i_{2k}}$

$$\begin{aligned}
\mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, \mathbb{P}_2^{\exists}, \ldots, \mathbb{G}) &\supseteq \bigcup_{i_2} \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, (\mathbb{P}_2^{i_2})^{\exists}, \ldots, \mathbb{G}) \\
&\supseteq \bigcup_{i_2} \mathcal{R}^-(\mathbb{X}^{\forall}, \mathbb{P}_1^{\forall}, (\check{\mathbb{P}}_2^{i_2})^{\exists}, \ldots, \mathbb{G}) \\
&\supseteq \bigcup_{i_{2k}} \bigcap_{i_{2k-1}} \mathcal{R}^-(\mathbb{X}^{\forall}, (\mathbb{P}_1^{i_1})^{\forall}, (\check{\mathbb{P}}_2^{i_2})^{\exists}, \ldots, \mathbb{G})
\end{aligned}$$